February 7, 2007

Exam Time ±2 Hours

NO CALCULATORS

1. Prove that $x^2 \quad y^2 \quad A^3$ always has integer solutions (x, y) whenever A is a

1. Since $x^2 \quad y^2 \quad (x \quad y)(x \quad y) \quad A^3 \qquad\qquad x \quad y \quad A^2$ and $x \quad y \quad A$

   Adding both of these equations, we obtain $2x = A^2 \quad A$ and $x \quad \frac{A(A \quad 1)}{2}$.

   Subtracting the two equations, we obtain $2y = A^2 \quad A$ and $y \quad \frac{A(A \quad 1)}{2}$.

   For any positive integer A, both A(A-1) and A(A+1) are products of 2 consecutive integers and are, therefore, both even. Hence, $x \quad \frac{A(A \quad 1)}{2}$ and $y \quad \frac{A(A \quad 1)}{2}$ are a pair of integer solutions to the equation.

2. The circumcenter of a triangle is the intersection of the perpendicular bisectors of its sides, namely, points E,F, G, and H. Any point on the perpendicular bisector of a segment is equidistant from the segment's endpoints. Since G is on the perpendicular bisector of $\overline{AD}$ and $\overline{DC}$, $\overline{AG} \quad \overline{DG} \quad \overline{CG}$. Since E is on the perpendicular bisector of $\overline{AB}$ and $\overline{BC}$, $\overline{AE} \quad \overline{EB} \quad \overline{EC}$. Since $\overline{AG} \quad \overline{CG}$ and $\overline{AE} \quad \overline{EC}$ points E and G are both equidistant from points A and C. Therefore, $\overline{EG}$ lies along the perpendicular bisector of $\overline{AC}$. Since point P lies on $\overline{EG}$, it is also on perpendicular bisector of $\overline{AC}$, making P equidistant from A and C. Thus $\overline{AP} \quad \overline{CP}$**.** In a similar manner**,** we can prove $\overline{BP} \quad \overline{DP}$.

3. Since *P*(a) = *P*(b) = *P*(c) = *P*(d) = 4, the polynomial *f*(x) = *P*(x)   4 has a, b, c, and d as zeros. Therefore, *f*(x) = *P*(x)   4 = (x   a)(x   b)(x   c)(x   d)*g*(x), where *g*(x) is a polynomial with integer coefficients.

Suppose there exists an integer m such that *P*(m) = 7. Then

f(m) = *P*(m)   4 = 7   4 = 3 = (m   a)(m

5. Let such a sequence be (m + 1), (m + 2), . . . , (m + k). The sum of the terms can be written as

$$km \quad \frac{k(k \quad 1)}{2} \quad \frac{k(2m \quad k \quad 1)}{2}.$$

Thus $\frac{k(2m \quad k \quad 1)}{2} \quad 2007 \quad 3^2 \; 223$ and $k(2m + k + 1) = 18\ 223$.

Since 223 is prime and is a factor of the product k(2m + k + 1), then 223 divides either k or 2m + k + 1. Since 1 + 2 + 3 + . . . + 63 = 2016 > 2007, then k < 63. Therefore, 223 divides 2m + k + 1 and k must be a factor of 18. Thus, k = 18, 9, 6, 3, 2, or 1.

If k = 18, then 2m + k + 1 = 223 implies m = 102. This gives the sequence
103, 104, 105, . . ., 120.

If k = 9, then 2m + k + 1 = 2 223 = 446, and m = 218. This gives the sequence
219, 220, 221, . . ., 227.

If k = 6, then 2m + 2t + 1 = 3 223 implies m = 331. This gives the sequence
332, 333, 334, . . , 337.

If k = 3, then 2m + k + 1 = 6 223 = 1338, and m = 667. This gives the sequence
668, 669, 670.

If k = 2, then 2m + 2t +1 = 9 223 implies m = 1002. This gives the sequence
1003, 1004.

If k = 1, then 2m + k + 1 = 18 223 = 4014 implies m = 2006. This gives the sequence with one number, 2007, which we are asked not to include.

Thus, there are five sequences of consecutive positive integers whose terms sum to 2007:

103, 104, 105, . . . , 120
219, 220, 221, . . . , 227
332, 333, 334, . . . , 337
668, 669, 670
1003, 1004