THE 2015–2016 KENNESAW STATE UNIVERSITY
HIGH SCHOOL MATHEMATICS COMPETITION

PART I – MULTIPLE CHOICE

For each of the following 25 questions, carefully blacken the appropriate box on the answer sheet with a #2 pencil. Do not fold, bend, or write stray marks on either side of the answer sheet. Each correct answer is worth 6 points. Two points are given if no box is marked. Zero points are given for an incorrect answer or if multiple boxes are marked. Note that wild guessing is likely to lower your score. When the exam is over, give your answer sheet to your proctor. You may keep your copy of the questions.

NO CALCULATORS

90 MINUTES

1. Between 1934 and 2015 there were 13 different presidents of the United States and 16 different vice presidents. If 11 of the vice presidents were never president, how many of the presidents were never vice president?

(A) 5 (B) 6 (C) 7 (D) 8 (E) 9

2. In the addition “

6. There are two four-digit numbers, each of the form A B C A, with the property that the two-digit number A B is a prime, the two-digit number B C is a square, and the two-digit number C A is the product of a prime and a square greater than 1. Compute the sum of these two four-digit numbers.

(A) 10,657 9.12 (t 0) sumour

13. Each integer from 1 to 9 is entered exactly once in the "cross-number" puzzle shown in such a way that the three-digit numbers appearing in 1-across, 2-across, 3-across, and 1-down are perfect squares. Compute the two-digit number appearing in 2-down.

(A) 76 (B) 58 (C) 52 (D) 36 (E) 12

14. The solutions to the equation $x^3 + ax^2 + bx + c = 0$ are three consecutive positive integers, compute the value of $\frac{{}^{2}}{+1}$

20. If $f(11) = 11$, and for all x, $f(x+3) = \dfrac{f(x) ! 1}{f(x)+1}$, compute $f(2015)$.

(A) 11 (B) $! \dfrac{1}{11}$ (C) $\dfrac{5}{6}$ (D) $! \dfrac{6}{5}$ (E) Ð11

21. Let A be a two-digit integer and let B be the integer obtained by reversing the digits of A. If $A^2 ! B^2$ is the square of an integer, compute $A^2 + B^2$.

(A) 4,941 (B) 5,265 (C) 5,913 (D) 6,885 (E) 7,361

22. Shown in the accompanying diagram is part of a regular polygon (ABCDEÉ) of unspecified number of sides $n > 4$. Another regular polygon having $\overline{!!"}$ as one side and angles ABD and EDB as consecutive angles is drawn. Which of the following is a possible value of n?

(A) 55

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Solutions

1. D Make a Venn diagram. Answer is 8.

The eight were: Franklin Roosevelt, Dwight Eisenhower, John Kennedy, Jimmy Carter, Ronald Reagan, Bill Clinton, George W. Bush, Barack Obama.

2. C Clearly, R = 0. The number F I V E will be smallest if F = 1. This means O must be 2, 3, or 4, and 2 makes the I smallest. Thus, I = 4. This leaves 3 and 5 for U and N (in either order), making V = 8. This leaves only 6, 7, or 9 for E. Therefore, the smallest value for the number F I V E is 1486.

3. E Let n = number of boys in the club at the start. Then $(n)(n) + 4(n + 4) = 301$. From which $n^2 + 4n - 285 = 0$. Factoring, $(n - 15)(n + 19) = 0$, and $n = 15$. Thus, there are $15 + 4 = 19$ boys now in the club.

4. C Method 1: The probability that the second number is the same as the first is 1/6. Therefore, 5/6 of the time, one die has a higher number than the other. By symmetry, the probability that the second die has the higher number is $(1/2)(5/6) = 5/12$.

6. B Since A B is prime, B must be 1, 3, 7, or 9. Since B C is a square and there are no squares in the 70Õs or 90Õs, B = 1 or 3, and this means C = 6. A quick check of the integers from 61 to 69 shows that 63 = (9)(7) and 68 = (4)(17) satisfy the third condition of the problem. Thus A = 3 or 8. Remembering the first condition, the only 4-digit numbers that work are 3163 and 8368. The required sum is 11,531

7. B Since PD = DC, ! PDC is isosceles and " CPD # " PCD.
Since ! DPB is also isosceles, " PDB # " PBD.
Let m" PCD = x. Representing angle measures as shown
in the diagram, m" BPD = 180 Ð 2(180 Ð 2x) = 4x Ð 180.
Therefore, m" APB = (180 Ð x) + (4x Ð 180) = 3x.
Hence, the required ratio is 1:3.

8. C It is easy enough to list all 10 possibilities: {0,1,2,8,9,10}, {0,1,3,7,9,10}, {0,1,4,6,9,10}
{0,2,3,7,8,10}, {0, 2,4,6,8,10}, {0,3,4,6,7,10}, {1,2,3,7,8,9}, {1,2,4,6,8,9}, {1,3,4,6,7,9},
and {2,3,4,6,7,8}.

9. A Let 3k and k denote the number of students in each subgroup and M denote the class mean. Then kM Ð 3kÐ

19. E Using the Law of Cosines on the triangle with sides 4, 6, 8,

$$8^2 = 4^2 + 6^2 - 2(4)(6)\cos\theta \quad \Rightarrow \quad \cos\theta = -\frac{1}{4}$$

Since the consecutive angles of a parallelogram are supplementary, the other angle of the parallelogram is $(180 - \theta)°$. Let d represent the length of the other diagonal. Now using the Law of Cosines on the triangle with sides 4, 6, d,

$$d^2 = 4^2 + 6^2 - 2(4)(6)[\cos(180 - \theta)] = 52 - 48(-\cos\theta) = 52 - 48\left(\frac{1}{4}\right) = 40 \text{ and } d = \sqrt{40}.$$

20. A $f(11) = 11$, $f(14) = \dfrac{f(11) - 1}{f(11) + 1} = \dfrac{5}{6}$, $f(17) = \dfrac{f(14) - 1}{f(14) + 1} = -\dfrac{1}{11}$,

$f(20) = \dfrac{f(17) - 1}{f(17) + 1} = -\dfrac{6}{5}$ and $f(23) = \dfrac{f(20) - 1}{f(20) + 1} = 11$.

Therefore, for all positive integers n, $f(11) = f(23) = f(35) = \ldots = f(11 + 12n)$.

Since $f(2015) = f(11 + 12 \cdot 167)$, then $f(2015) = 11$.

21. E Let $A = 10x + y$ and

23. E Expand 101^{20} using the binomial theorem:

$$(100+1)^{20} = 1 + (20)(100) + \binom{20}{2}(100^2) + \binom{20}{3}(100^3) + \binom{20}{4}(100^4) + \ldots$$

The first 3 terms will not affect the ninth digit from the right since each has fewer than 8 digits. None of the omitted terms at the end will affect the ninth digit from right since each has more than 9 terminal zeros. The fourth and fifth terms are 1,140,000,000 and 484,500,000,000, so that the ninth digit from the right in their sum is 5 + 1 = 6.

24. D $\log_{\sin x}(\tan x) = \log_{\tan x}(\sin x) = \dfrac{1}{\log_{\sin x}(\tan x)}$ using the change of base formula,

where $\sin x > 0$ and $\tan x > 0$.

Therefore, $\left[\log_{\sin x}(\tan x)\right]^2 = 1$, which implies $\log_{\sin x}(\tan x) = \pm 1$.

$\log_{\sin x}(\tan x) = 1$ & $\tan x = \sin x$ & $\cos x = 1$. However, this would make $\sin x = 0$.

$\log_{\sin x}(\tan x) = -1$ & $\cos x = \sin^2 x = 1 - \cos^2 x$

Therefore, $\cos^2 x + \cos x - 1 = 0$ & $\cos x = \dfrac{-1 \pm \sqrt{5}}{2}$.

Since $\dfrac{-1 - \sqrt{5}}{2} < -1$, the only possible value of $\cos x$ is $\dfrac{-1 + \sqrt{5}}{2}$.

C′ (0,5)

C (3,4)

B′

B (3,0)

25. A Since ABC is a right triangle, AC = AC′ = 5. Let D be the point of intersection of $\overline{}$ and $\overline{B'C'}$, as shown in the diagram. Since BC is parallel to the y-axis, ∠C ≅ ∠CAC′. Therefore, ∠B′C′A ≅ ∠CAC′, so that △ADC′ is isosceles. Thus, the altitude from D meets $\overline{}$ at its midpoint, M, and C′M = 2.5.

Since △MDC′ and △BAC are both right triangles with ∠C ≅ ∠C′, the triangles are similar. Thus, $\dfrac{MC'}{BC} = \dfrac{C'D}{AC}$. Substituting, $\dfrac{2.5}{4} = \dfrac{C'D}{5}$, from which $C'D = \dfrac{25}{8}$.

Hence, $DB' = 4 - \dfrac{25}{8} = \dfrac{7}{8}$, and the area of right triangle DAB′ is

$$\frac{1}{2}(AB')(B'D) = \frac{1}{2}(3)\left(\frac{7}{8}\right) = \frac{21}{16}$$