THE 2018-2019 KENNESAW STATE UNIVERSITY HIGH SCHOOL MATHEMATICS COMPETITION PART II

Calculators are NOT permitted

Time allowed: 2 hours

1. Let *m* be a three-digit integer with distinct digits. Find all such integers *m* which are equal to the average (arithmetic mean) of the six numbers obtained by forming all possible arrangements of the digits of *m*. Prove that you have found them all.

2. A bag contains *N* balls, some of which are red and the rest yellow. Two balls are drawn randomly from the bag, without replacement. If the probability that the two balls are the same color is equal to the probability that they are different colors, compute, with proof, the set of all possible values of *N*.

3. Let $P(x)$ b $P(0)$ is an odd integer and that $P(1)$ is also an odd integer. Prove that $P(c)$ is an odd integer for all integers *c*.

4. Let S = a set of consecutive positive integers beginning with 1. All subsets of *S* that do not contain two consecutive numbers are formed. The product of the elements in each subset is calculated. (Note: if a subset contains only one number, the product of its elements is the number itself.) Let N = the sum of the squares of all these products.

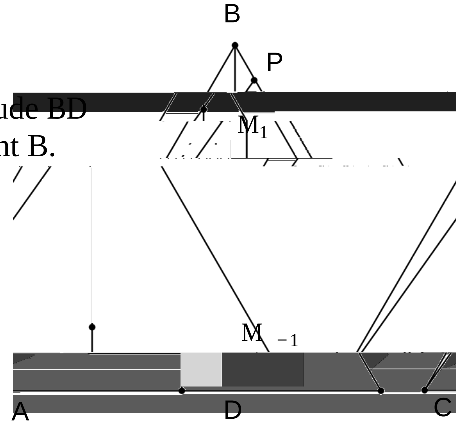
   For example, if $S = \{1, 2, 3\}$, then the allowable subsets are $\{1\}$, $\{2\}$, $\{3\}$, and $\{1, 3\}$. The products are 1, 2, 3, and 3, and $N = 1^2 + 2^2 + 3^2 + 3^2 = 23$.

   Find, with proof, the value of *N*

5. In equilateral ΔABC, points M_1, M_2, M_3 M $_{-1}$ divide altitude BD into *n* segments of equal length ($n > 1$), with M_1 closest to point B.

   AM_1 is extended to meet side BC at point P.

   Prove that $\frac{AM_1}{M_1P} = 2n - 1$.



Solutions

1. $m = 100a + 10b + c$. If we arrange the digits of m in all six possible ways, then each of a, b, and c will occur exactly twice in the 1's place, twice in the 10's place, and twice in the 100's place. Thus, the sum of the six arrangements is

$$2(a + b + c)\,(100 + 10 + 1) = 222(a + b + c).$$

Hence, the arithmetic mean of these six numbers is $37(a + b + c)$.

$a + 10b + c = 37(a + b + c)$, we have $7a = 3b + 4c$.

Method 1: $7a = 3b + 4c \;\Rightarrow\; 3b + 4c$ is a multiple of 7. Make a chart, remembering that all three digits are distinct. The values of c in the middle column of the chart are the only ones for which $3b + 4c$ is a multiple of 7.

Therefore, there are a total of six values of m: 370, 407, 481, 518, 592, and

Method 2: $7a = 3b + 4c \;\Rightarrow\; 7(a – c) = 3(b – c)$.

Now, $-9 \le b – c \le 9$ and 7 divides $b – c$. There are just three possibilities.

If $b – c = 0$, then $a = b = c$, which cannot be since the digits of m are all different.

If $b – c = 7$, then $a – c = 3$. In this case $b = c + 7$ and $a = c + 3$, leaving only three possibilities for c, namely 0, 1, and 2. These yield m = 370, 481, and 592.

Finally, if $b – c = –7$, then $a – c = –3$. In this case $b = c – 7$ and $a = c –$

2. Let K = the number of red balls and $N - K$

4. To get an idea, we first look at simpler versions of the problem.

Consider the set {1, 2}. The allowable subsets are {1}, {2}. The products are 1, 2, and the sum of their squares is 5.

For the set {1, 2, 3}, we already know the sum of the squares is 23.

Consider the set {1, 2, 3, 4}. The allowable subsets are {1}, {2}, {3}, {4}, {1, 3}, {2, 4} and {1, 4}. The products are 1, 2, 3, 4, 3, 8, 4, and the sum of their squares is 119.

Consider the set {1, 2, 3, 4, 5}. The allowable subsets are {1}, {2}, {3}, {4}, {5}, {1, 3}, {1, 4}, {1, 5}, {2, 4}, {2, 5}, {3, 5} and {1, 3, 5}. The products are 1, 2, 3, 4, 5, 3, 4, 5, 8, 10, 15, and 15, and the sum of their squares is 719.

Observe that the sum of the squares of the products appears to be $(n + 1)! - 1$, where n is the number of elements in the set S. We prove that this expression is correct by mathematical induction.

For $n = 1$, the formula is trivially true.

Assume that for a set consisting of the first K positive integers, the desired result is $(K + 1)! - 1$.

Consider the set consisting of the first $(K + 1)$ posit
subsets with no consecutive numbers into two subcollections: subsets containing $K + 1$, and those that don’t.

Each subset in the first subcollection can be represented as the union of $\{K + 1\}$ and the non- K–1} which do not contain consecutive numbers. Therefore, by the induction hypothesis, the sum of the squares of the products from this first subcollection is $(\quad + 1)^2(\quad ! - 1) + (\quad + 1)^2$ the products from the second subcollection is $(K + 1)! - 1$.

Adding $(\quad + 1)^2(\quad ! - 1) + (\quad + 1)^2$ and $(K + 1)! - 1$ and simplifying, we obtain $(K + 2)! - 1$. Therefore, the desired value is $17! - 1$.

5. <u>Method 1</u>

Let AB = BC = *a* and BP = *y*

Then the length of altitude BD is $\frac{\sqrt{3}}{2}$ and the length of BM_1 is $\frac{\sqrt{3}}{2}$

BM_1 is an angle bisector in ΔABP,

$\frac{AM_1}{M_1P} = \frac{AB}{BP} = -.$

The area of $\Delta ABM_1 = \frac{1}{2}(AB)(BM_1)\sin 30 = \frac{1}{2}(\quad)BM_1 \;\frac{1}{2} = \frac{1}{4}(\quad)BM_1.$

rea of $\Delta BM_1P = \frac{1}{2}(BP)(BM_1)\sin 30 = \frac{1}{2}(\quad)BM_1 \;\frac{1}{2} = \frac{1}{4}(\quad)BM_1$

and the area of $\Delta ABP = = \frac{1}{2}(AB)(BP) \qquad \frac{1}{2}\; y\frac{\sqrt{3}}{2} = \frac{1}{4} \quad \sqrt{3}.$

B