

Solutions

1. Method 1: Represent the four numbers in arithmetic sequence as $a, a + d, a + 2d$, and $a + 3d$. Then, the geometric sequence is $a, a + d, a + 2d$, and $a + 3d + 12$. Therefore,

$$\frac{a + d}{a} = \frac{a + 2d}{a + d + 12} \quad (1)$$

Similarly, $\frac{a + 2d}{a + d} = \frac{a + 3d + 12}{a + 2d}$.

Therefore, $a = d + 9$.
Substituting this last equation into (1) and simplifying,

$$d = 15, d = -3.$$

If $d = 15$, $a = 24$, and the arithmetic sequence is 24, 39, 54, 69

If $d = -3$, $a = 6$, and the arithmetic sequence is 6, 3, 0, -3.

A quick check shows that 24, 39, 54, 69 satisfies the conditions of the problem, with the corresponding geometric sequence being 24, 36, 54, 81.

However, 6, 3, 0, -3 does not work since 6, 0, 0, 9 is not a geometric sequence.

Therefore, the only arithmetic sequence is 24, 39, 54, 69.

Method 2: Represent the four numbers in arithmetic sequence as $a, a + d, a + 2d$, and $a + 3d$. Let the terms of the geometric sequence be represented by a, ar, ar^2, ar^3 . Then

$$(1) ar = a + d \quad (2) ar^2 = a + 2d \quad (3) ar^3 = a + 3d + 12$$

$$\text{From (1) } a(r - 1) = d \quad \text{From (2) } a(r^2 - r) = 2d \quad \text{From (3) } a(r^3 - r^2) = 3d + 12$$

$$\text{Therefore, } (d - 3)(r + 1) = 2d \quad (4) \quad r + 1 = \frac{2d}{d - 3}$$

$$\text{From (3) } ar^3 = 3d + 12 \quad a(r^3 - r^2) = 3d + 12 \quad a(r - 1)(r^2 + r + 1) = 3(d + 4)$$

Substituting (1) into this last equation and dividing by $d - 3$, we obtain

$$(5) \quad (r^2 + r + 1) = \frac{3(d + 4)}{d - 3}$$

$$\text{Substituting (4) into (5) we obtain } \frac{2d}{d - 3} + \frac{2d}{d - 3} + 1 = \frac{3(d + 4)}{d - 3}$$

$$\text{From (4) } r = \frac{2d}{d - 3} - 1 = \frac{2d - d + 3}{d - 3} = \frac{d + 3}{d - 3}$$

Therefore, $\frac{d + 3}{d - 3} + \frac{d + 3}{d - 3} + 1 = \frac{3(d + 4)}{d - 3}$ from which we eventually obtain $d = 15$. Thus, from (4),

$r = \frac{18}{12} = \frac{3}{2}$ and from (1) $a = 24$. Therefore, the only such arithmetic sequence is 24, 39, 54, 69.

2. Assume that $f(x) = 0$ has an integer root a . Since the lead coefficient of $f(x)$ is 1, the sum of the roots is $-a$. Since a is an integer, $f(x)$ has another integer root $-a$. Thus, $f(x) = (x - a)(x + a)$, and $f(300) = (300 - a)(300 + a)$. Without loss of generality, let $a > 0$.

Since we are given $f(300)$ is prime, this means that $(300 - a) = 1$ and $(300 + a)$ is prime. Therefore, $a = 299$ while $300 + a = 599$ (since 293 and 307 are the closest primes to 300). Since the product of the roots of $f(x) = 0$ is $-a^2$, $-a^2 = -299^2 = -89401$. But this is a contradiction, since we are given $f(300) = 2093$.

Therefore, $f(x) = 0$ has no integer solutions.

3. Method 1: Construct diagonal AC . Since $\triangle ADC \cong \triangle ABC$ (SSS), $\angle ADC = \angle ABC$. Therefore, $m\angle DCA = m\angle BCA = \frac{1}{2} m\angle ABC$. Let $m\angle BCA = x$ and $m\angle ABC = 2x$, and let $AC = a$. Using the Law of Sines on $\triangle ABC$,

$$\frac{a}{\sin 2x} = \frac{BC}{\sin x} \implies BC = a \cos x$$

Using the Law of Cosines on $\triangle ABC$,

$$a^2 = BC^2 + AC^2 - 2 \cdot BC \cdot AC \cdot \cos 2x \implies a^2 = a^2 \cos^2 x + a^2 - 2a^2 \cos x \implies \cos x = \frac{1}{2}$$

Now, construct the altitude of $\triangle PBC$ to BC , meeting BC at point M . Since $\triangle PBC$ is isosceles ($\angle C = \angle B$), M is the midpoint of BC . Thus, $BM = MC = 2.5$.

Then, in right $\triangle PMB$, $\cos B = \frac{BM}{PB} = \frac{1}{2}$, and $PB = 20$.

Finally, using the Pythagorean Theorem

on $\triangle PMB$,

$$\text{and } PM = \sqrt{PB^2 - BM^2} = \sqrt{20^2 - 2.5^2} = \sqrt{397.5}$$

which is the desired distance.

Method 2: Construct diagonal AD . Since $\triangle ADB$ and $\triangle CDB$ are both isosceles triangles, $\triangle ADC \cong \triangle ABC$ and both are congruent to $\triangle BCD$. Thus, $\triangle PBC$ is isosceles. Let $PB = x$, $PA = x - 4$, and $PD = x - 5$.

Using the Law of Cosines on $\triangle PAD$,

$$(1) \quad 16 =$$

Using the Law of Cosines on $\triangle PBC$,

$$25 = \dots \cos P = \dots$$

Substituting into (1) above,

$$\dots$$

Carefully simplifying this last equation, we obtain

Factoring, \dots from which $x = -$ (impossible) and $x = 20$.

Finally, construct the altitude of PM of $\triangle PBC$ and noting that M is the midpoint of BC , use the Pythagorean Theorem on $\triangle PMB$.

$$\text{and } PM = \dots, \text{ or } \dots,$$

which is the desired distance.

4. Assume that \dots for some positive integer a .

We first prove that n is not a multiple of p .

Suppose that \dots for some integer k . Then \dots and, therefore,

Hence, p must divide a which means \dots is an integer, and \dots .

Then, $k < \dots < k + 1$, which is impossible. Therefore, n is not a multiple of p .

Next, we prove that n and $n + p$ have no common prime factors. Suppose a prime q divides both n and $n +$

5. The desired ratio is $\frac{1}{2}$.

Method 1

Construct DE and DF . Represent the area of $\triangle ABC$ as $[ABC]$.

$[AFE] = [AFD]$, since $DF = FE$, and $\triangle AFE$ and $\triangle AFD$ have the same altitude from point A . Similarly, $[BFE] = [BFD]$.

Thus, $[AEB] = 2[AFB]$,

$[AFE] = 2[EFC]$, since $AE = 2(EC)$ and $\triangle AFE$ and $\triangle EFC$ have the same altitude from point F . Similarly, $[BFD] = 2[AFD] = 2[AFE] = 4[EFC]$.

Also, $[ADE] = [AFD] + [AFE] = 4[EFC]$.

$[AEB] = \frac{1}{2}[ABC]$, since $AE = \frac{1}{2}AC$ and the triangles have the same altitude from point B .

Therefore, $[AEB] = 2[AFB] = \frac{1}{2}[ABC] \implies [AFB] = \frac{1}{4}[ABC]$

Also, $[AFB] = [AFD] + [BFD] = [AFE] + [BFD]$
 $= 2[EFC] + 2[AFD] = 2[EFC] + 4[EFC] = 6[EFC]$.

Therefore, $[AFB] = \frac{1}{4}[ABC] = 6[EFC] \implies [EFC] = \frac{1}{24}[ABC]$.

Finally, $[BFC] = [ABC] - [AEB] - [ADE] - [BFD]$

$= [ABC] - \frac{1}{4}[ABC] - \frac{1}{4}[ABC] - \frac{1}{4}[ABC] = \frac{1}{4}[ABC]$.

Method 2

Construct perpendiculars from D , A , F , and E to BC , and label the points of intersection G , H , I , and J , respectively.

The area of $\triangle ABC = \frac{1}{2}BC \cdot AH$

Since DG is parallel to AH , $\triangle DGB$ is similar to $\triangle AHB$.

Therefore, $\frac{DG}{AH} = \frac{DB}{AB}$

Method 3

Let $EC = x$, $EA = 2x$, $AD = y$, $BD = 2y$, and $DF = EF = w$.

Let $\angle ADE = \theta$ and $\angle AED = \phi$.

$$\text{Area } \triangle ABC = \frac{1}{2} \cdot AC \cdot BC \cdot \sin \angle C = \frac{1}{2} \cdot 3x \cdot 2y \cdot \sin \theta = 3xy \sin \theta.$$

$$\text{Area } \triangle AED = \frac{1}{2} \cdot EA \cdot ED \cdot \sin \angle AED = \frac{1}{2} \cdot 2x \cdot w \cdot \sin \phi = xw \sin \phi.$$

$$\text{Area } \triangle EFC = \frac{1}{2} \cdot EC \cdot EF \cdot \sin \angle CEF = \frac{1}{2} \cdot x \cdot w \cdot \sin \theta = \frac{1}{2} xw \sin \theta.$$

$$\text{Area } \triangle FDB = \frac{1}{2} \cdot FD \cdot BD \cdot \sin \angle FDB = \frac{1}{2} \cdot w \cdot 2y \cdot \sin \phi = wy \sin \phi.$$

Using the Law of Sines on $\triangle AED$,