1. Each of three cards has an integer written on it. The three integers p, q, r satisfy the condition $0 \leq p < q < r$. Three players A, B, C mix the cards and pick one each. They record the number on their card, mix the cards and pick one each again. The number on the card they select is added to their previous number. This process is repeated at most ten times, after which A has 20 points, B has 10 points, and C has 9 points. If B got the r card in the last round, determine, with proof,

<u>Solutions</u>

1. Because a constant number of points, $p + q + r$, is awarded in each round, $p + q + r$ must divide the total number of points, 39.

(b) Since AOD   BCO, $\frac{AD}{OB} = \frac{AO}{BC}$, or (AD)(BC) = (AO)(OB). Since AO = OB = ½ (AB), we get $(AB)^2 = 4(AD)(BC)$.

4. Suppose we could write a number both as $n(n + 1)$ and as $m(m + 1)(m + 2)(m + 3)$, where n and m are positive integers. Multiplying the outer and inner factors of the second expression gives $(\ ^2 + 3\ )(\ ^2 + 3\ + 2)$. Letting $k = \ ^2 + 3\ $, we have $\ (\ + 1) = \ (\ + 2)$. Adding 1 to each side, $\ ^2 + \ + 1 = \ ^2 + 2\ + 1 = (\ + 1)^2$. So $\ ^2 + \ + 1$ is a perfect square. But $\ ^2 < \ ^2 + \ + 1 < (\ + 1)^2 = \ ^2 + 2\ + 1$. Thus, $\ ^2 + \ + 1$ is a perfect square that lies between 2 consecutive perfect squares, which is impossible. Therefore, there are no integers that can be written both as the product of two and also four consecutive positive integers.

5. <u>Method 1</u>

In ABC, since $\overline{CD}$ bisects ACB, $\frac{10}{AD} = \frac{5}{6-AD}$,
from which AD = 4 and DB = 2.

Using the Law of Cosines on ABC,

$$5^2 = 10^2 + 6^2 - 2(10)(6)\cos A \qquad \cos A = \frac{37}{40}.$$

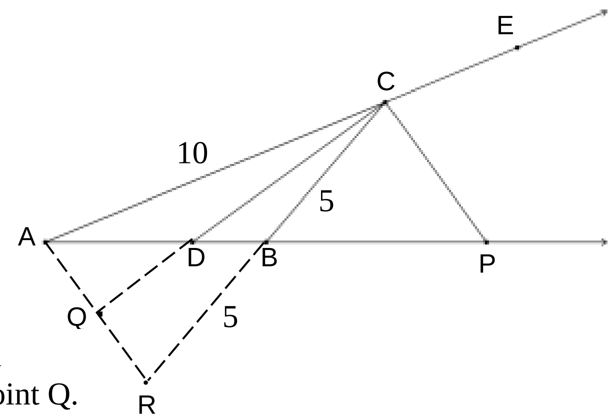
Using the Law of Cosines on ADC,

$$CD^2 = 10^2 + 4^2 - 2(10)(4)\frac{37}{40} \qquad CD = \sqrt{42}.$$

Using the Law of Cosines on ADC again,

$$10^2 = 4^2 + (\sqrt{42})^2 - 2(4)(\sqrt{42})\cos\ ADC \qquad \cos\ ADC = \frac{-\sqrt{42}}{8}.$$

Since CDP is supplementary to ADC,



Method 2

From Method 1, AD = 4 and DB = 2, and DCP is a right angle.

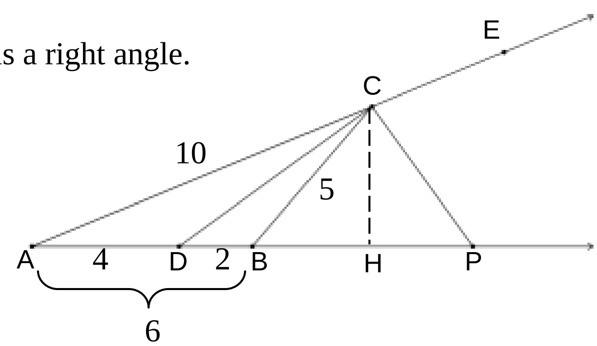
Extend $\overline{CB}$ its own length through B to a point R, and construct $\overline{AR}$. Extend $\overline{CD}$ through D to meet $\overline{AR}$ at point Q.

Since CR = 10, ACR is isosceles, and since $\overline{CQ}$ bisects vertex angle ACR, $\overline{CQ}$ is also an altitude in ACR. Then $\overline{AR}$ is parallel to $\overline{CP}$ since $\overline{CQ}$ is perpendicular to both. Therefore, BCP and BRA are congruent alternate interior angles.

Then, ABR PBC, so that BP = AB = 6, and DP = DB + BP = 8.

Method 3

From Method 1, AD = 4 and DB = 2, and DCP is a right angle.



Construct the altitude of ABC from C, meeting Line AB at point H. Let BH = and CH = .

Using the Pythagorean Theorem on ACH,
$(\ + 6)^2 + \ ^2 = 10^2 \qquad ^2 = 64 - 12 \ - \ ^2$

Using the Pythagorean Theorem on DCH,
$(\ + 2)^2 + \ ^2 = 5^2 \qquad ^2 = 21 - 4 \ - \ ^2$.

Setting the two equations equal and solving for , we get $= \frac{13}{4}$, from which $= \frac{\sqrt{231}}{4}$.
Using the Pythagorean Theorem on CDH, CD = $\sqrt{42}$.

Because right DCP and right DHC share D, they are similar. Then
$\frac{DH}{DC} = \frac{DC}{DP} \qquad DP = \frac{(DC)^2}{DH} = \frac{42}{\frac{21}{4}} = 8.$